

HIGHER IDENTITIES FOR THE TERNARY COMMUTATOR

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ABSTRACT. We use computer algebra to study polynomial identities for the trilinear operation $[a, b, c] = abc - acb - bac + bca + cab - cba$ in the free associative algebra. It is known that $[a, b, c]$ satisfies the alternating property in degree 3, no new identities in degree 5, a multilinear identity in degree 7 which alternates in 6 arguments, and no new identities in degree 9. We use the representation theory of the symmetric group to demonstrate the existence of new identities in degree 11. The only irreducible representations of dimension < 400 with new identities correspond to partitions $2^5 1$ and $2^4 1^3$ and have dimensions 132 and 165. We construct an explicit new multilinear identity for partition $2^5 1$ and we demonstrate the existence of a new non-multilinear identity in which the underlying variables are permutations of $a^2 b^2 c^2 d^2 e^2 f$.

1. INTRODUCTION

The theory of multioperator algebras (Ω -algebras), by which is meant vector spaces with multilinear operations, was first studied systematically by the school of Kurosh in Moscow; see [1, 17]. In particular, a natural generalization of the Lie bracket to the n -ary setting is the alternating n -ary sum (n -commutator):

$$[a_1, \dots, a_n] = \sum_{\sigma \in S_n} \epsilon(\sigma) a_{\sigma(1)} \cdots a_{\sigma(n)},$$

where $\epsilon(\sigma)$ is the sign of the permutation σ . This operation provides unexpected algebraic structures on vector fields [11, 12], and plays an essential role in the construction of universal enveloping algebras of Filippov algebras (n -Lie algebras) [13]. For many other applications, especially to theoretical physics, see the recent survey of n -ary analogues of Lie algebras [9].

For $n = 3$, the alternating ternary sum (ternary commutator) has the form

$$[a, b, c] = abc - acb - bac + bca + cab - cba.$$

The first explicit polynomial identity which is satisfied by this operation in the free associative algebra, but which does not follow from the alternating property in degree 3, was found in 1998; see [2]. This identity has degree 7:

$$\sum_{\sigma \in S_6} \epsilon(\sigma) ([[[b^\sigma, c^\sigma, d^\sigma], a, e^\sigma], f^\sigma, g^\sigma] + [[a, b^\sigma, c^\sigma], [d^\sigma, e^\sigma, f^\sigma], g^\sigma]) \equiv 0.$$

Two years later, it was shown that there are no new identities in degree 9; see [3]. Ten years later, the identity in degree 7 was rediscovered [10], and was generalized

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to all odd n [8]; the situation is much simpler for even n [15, 16, 18]. For identities relating the ordinary and ternary commutators, see [14]. For the partially alternating ternary sum in an associative dialgebra (Loday algebra), see [6].

In this paper, we use computer algebra to show that further new polynomial identities for the ternary commutator exist in degree 11. We construct an explicit new multilinear identity for partition $2^5 1$ and we demonstrate the existence of a new non-multilinear identity in which the underlying variables are permutations of $a^2 b^2 c^2 d^2 e^2 f$.

Owing to the large size of the matrices involved in our computations, we used modular arithmetic to save memory. By choosing a suitable modulus, we found it easy to perform “rational reconstruction” of the correct results in characteristic 0 from the results obtained in characteristic p . Underlying all of these computations is the structure theory of the group algebra of the symmetric group S_n , which is semisimple both in characteristic 0 and in characteristic $p > n$. For further information, see [4, Lemma 8] and [5, §5.5].

2. PRELIMINARIES

For an alternating trilinear operation, every monomial in degree 11 can be written in terms of one of the following eight association types (placements of brackets); we display these types with the identity permutation of the arguments:

$$(*) \quad \left\{ \begin{array}{ll} 1: [[[[[a, b, c], d, e], f, g], h, i], j, k] & 2: [[[[[a, b, c], [d, e, f], g], h, i], j, k] \\ 3: [[[[[a, b, c], d, e], [f, g, h], i], j, k] & 4: [[[[[a, b, c], [d, e, f], [g, h, i]], j, k] \\ 5: [[[[[a, b, c], d, e], f, g], [h, i, j], k] & 6: [[[[[a, b, c], [d, e, f], g], [h, i, j], k] \\ 7: [[[[[a, b, c], d, e], [f, g, h], i, j], k] & 8: [[[[[a, b, c], d, e], [f, g, h], [i, j, k]] \end{array} \right.$$

The number of multilinear monomials in each type can be easily calculated using the alternating property of $[a, b, c]$; the total is

$$\frac{11!}{6^1 2^4} + \frac{1}{2} \cdot \frac{11!}{6^2 2^2} + \frac{11!}{6^2 2^2} + \frac{1}{6} \cdot \frac{11!}{6^3 2} + \frac{11!}{6^2 2^2} + \frac{1}{2} \cdot \frac{11!}{6^3} + \frac{1}{2} \cdot \frac{11!}{6^2 2^2} + \frac{1}{2} \cdot \frac{11!}{6^3 2} = 1401400.$$

Since this number is so large, we cannot process all the monomials at once, so we use the representation theory of the symmetric group S_{11} to decompose the problem into a sequence of smaller problems, each corresponding to an irreducible representation. (For a detailed discussion of this approach, see [4, §4] or [5, §5].)

Using representation theory requires that we enumerate the symmetries of the association types (*); each symmetry is a two-term identity expressing the fact that the value of a monomial changes sign after a transposition of two factors. Since the symmetric group S_3 is generated by the transpositions (12) and (23), every symmetry is a consequence of the 43 symmetries corresponding to the monomials π in Table 1. In that table, π represents the identity $\iota + \pi \equiv 0$, where ι represents the monomial with the identity permutation of the variables in the same association type. These symmetries are the consequences in degree 11 of the alternating properties $[a, b, c] + [b, a, c] \equiv 0$ and $[a, b, c] + [a, c, b] \equiv 0$ in degree 3.

We also need to determine the consequences in degree 11 of the known polynomial identity in degree 7; see [2, 8, 10]. We write this identity symbolically as

1	$\left\{ \begin{array}{l} [[[[[b, a, c], d, e], f, g], h, i], j, k] \\ [[[[[a, c, b], d, e], f, g], h, i], j, k] \\ [[[[[a, b, c], e, d], f, g], h, i], j, k] \\ [[[[[a, b, c], d, e], g, f], h, i], j, k] \\ [[[[[a, b, c], d, e], f, g], i, h], j, k] \\ [[[[[a, b, c], d, e], f, g], h, i], k, j] \end{array} \right.$	2	$\left\{ \begin{array}{l} [[[[[b, a, c], [d, e, f], g], h, i], j, k] \\ [[[[[a, c, b], [d, e, f], g], h, i], j, k] \\ [[[[[d, e, f], [a, b, c], g], h, i], j, k] \\ [[[[[a, b, c], [d, e, f], g], i, h], j, k] \\ [[[[[a, b, c], [d, e, f], g], h, i], k, j] \end{array} \right.$
3	$\left\{ \begin{array}{l} [[[[[b, a, c], d, e], [f, g, h], i], j, k] \\ [[[[[a, c, b], d, e], [f, g, h], i], j, k] \\ [[[[[a, b, c], e, d], [f, g, h], i], j, k] \\ [[[[[a, b, c], d, e], [g, f, h], i], j, k] \\ [[[[[a, b, c], d, e], [f, h, g], i], j, k] \\ [[[[[a, b, c], d, e], [f, g, h], i], k, j] \end{array} \right.$	4	$\left\{ \begin{array}{l} [[[[[b, a, c], [d, e, f], [g, h, i]], j, k] \\ [[[[[a, c, b], [d, e, f], [g, h, i]], j, k] \\ [[[[[d, e, f], [a, b, c], [g, h, i]], j, k] \\ [[[[[a, b, c], [g, h, i], [d, e, f]], j, k] \\ [[[[[a, b, c], [d, e, f], [g, h, i]], k, j] \end{array} \right.$
5	$\left\{ \begin{array}{l} [[[[[b, a, c], d, e], f, g], [h, i, j], k] \\ [[[[[a, c, b], d, e], f, g], [h, i, j], k] \\ [[[[[a, b, c], e, d], f, g], [h, i, j], k] \\ [[[[[a, b, c], d, e], g, f], [h, i, j], k] \\ [[[[[a, b, c], d, e], f, g], [i, h, j], k] \\ [[[[[a, b, c], d, e], f, g], [h, j, i], k] \end{array} \right.$	6	$\left\{ \begin{array}{l} [[[[[b, a, c], [d, e, f], g], [h, i, j], k] \\ [[[[[a, c, b], [d, e, f], g], [h, i, j], k] \\ [[[[[d, e, f], [a, b, c], g], [h, i, j], k] \\ [[[[[a, b, c], [d, e, f], g], [i, h, j], k] \\ [[[[[a, b, c], [d, e, f], g], [h, j, i], k] \end{array} \right.$
7	$\left\{ \begin{array}{l} [[[[[b, a, c], d, e], [[f, g, h], i, j], k] \\ [[[[[a, c, b], d, e], [[f, g, h], i, j], k] \\ [[[[[a, b, c], e, d], [[f, g, h], i, j], k] \\ [[[[[f, g, h], i, j], [[a, b, c], d, e], k] \end{array} \right.$	8	$\left\{ \begin{array}{l} [[[[[b, a, c], d, e], [f, g, h], [i, j], k] \\ [[[[[a, c, b], d, e], [f, g, h], [i, j], k] \\ [[[[[a, b, c], e, d], [f, g, h], [i, j], k] \\ [[[[[a, b, c], d, e], [g, f, h], [i, j], k] \\ [[[[[a, b, c], d, e], [f, h, g], [i, j], k] \\ [[[[[a, b, c], d, e], [i, j, k], [f, g, h]] \end{array} \right.$

TABLE 1. The symmetries of the association types in degree 11

$I(a, b, c, d, e, f, g) \equiv 0$, where

$$I(a, b, c, d, e, f, g) = \sum_{\sigma \in S_6} \epsilon(\sigma) ([[[[b^\sigma, c^\sigma, d^\sigma], a, e^\sigma], f^\sigma, g^\sigma] + [[a, b^\sigma, c^\sigma], [d^\sigma, e^\sigma, f^\sigma], g^\sigma]).$$

We collect similar terms in this identity using the alternating property of the ternary commutator, and see that the total number of distinct terms is $\binom{6}{3,1,2} + \binom{6}{2,3,1} = 60 + 60 = 120$. From the alternating property of $[a, b, c]$, and the alternating property of $I(a, b, c, d, e, f, g)$ in the arguments b, \dots, g , it follows that every consequence of $I(a, b, c, d, e, f, g) \equiv 0$ in degree 9 is a linear combination of permutations of three identities, the first two obtained by substituting a triple for a variable, and the third obtained by embedding the identity in a triple:

$$\begin{aligned} I([a, h, i], b, c, d, e, f, g) &\equiv 0, & I(a, [b, h, i], c, d, e, f, g) &\equiv 0, \\ I(a, b, c, d, e, f, g), h, i &\equiv 0. \end{aligned}$$

Similarly, every consequence of these three identities in degree 11 is a linear combination of permutations of the eight identities in Table 2. (It is a coincidence that the number of association types is equal to the number of consequences of

- 1: $I([a, j, k], h, i, b, c, d, e, f, g) \equiv 0$, 2: $I([a, h, i], [b, j, k], c, d, e, f, g) \equiv 0$,
 3: $I([a, h, i], b, c, d, e, f, g, j, k) \equiv 0$, 4: $I(a, [[b, j, k], h, i], c, d, e, f, g) \equiv 0$,
 5: $I(a, [b, h, i], [c, j, k], d, e, f, g) \equiv 0$, 6: $[I(a, [b, h, i], c, d, e, f, g), j, k] \equiv 0$,
 7: $[I(a, b, c, d, e, f, g), [h, j, k], i] \equiv 0$, 8: $[[I(a, b, c, d, e, f, g), h, i], j, k] \equiv 0$.

TABLE 2. The consequences of $I(a, b, c, d, e, f, g) \equiv 0$ in degree 11

$I(a, b, c, d, e, f, g)$.) We call these consequences the “liftings” of $I(a, b, c, d, e, f, g)$ to degree 11. We summarize this discussion in the following lemma.

Lemma 2.1. *Every polynomial identity in degree 11 satisfied by the ternary commutator, which is a consequence of identities of lower degree, is a linear combination of permutations of the identities in Tables 1 and 2.*

3. NEW IDENTITIES IN DEGREE 11

Let λ be a partition of 11 with k parts; we write

$$\lambda = (\lambda_1, \dots, \lambda_k), \quad \lambda_1 \geq \dots \geq \lambda_k, \quad \lambda_1 + \dots + \lambda_k = 11.$$

Let d_λ be the dimension of the corresponding irreducible representation of S_{11} . For any permutation $\pi \in S_{11}$ the $d_\lambda \times d_\lambda$ matrix R_π^λ representing π in the natural representation can be computed using the methods of [7]; see also [5, §5]. Taking linear combinations gives the matrix representing any element of the group algebra $\mathbb{Q}S_{11}$ over the rational field \mathbb{Q} . This provides an algorithm for explicit computation of the isomorphism ϕ from $\mathbb{Q}S_{11}$ to its Wedderburn decomposition, by which we mean the direct sum over all partitions λ of matrix algebras of size $d_\lambda \times d_\lambda$. Any multilinear polynomial P of degree 11 in the ternary commutator can be expressed as a sum of eight elements of $\mathbb{Q}S_{11}$, one term for each association type $(*)$. For each partition λ , the projection of P to the corresponding component of the Wedderburn decomposition consists of an ordered list of eight $d_\lambda \times d_\lambda$ matrices. The partitions λ for which $d_\lambda < 400$ are given in Table 3.

We apply this discussion to the symmetries of the association types in Table 1 and the consequences of $I(a, b, c, d, e, f, g)$ in Table 2. We first construct a $43d_\lambda \times 8d_\lambda$ matrix consisting of $d_\lambda \times d_\lambda$ blocks: for $i = 1, \dots, 43$ and $j = 1, \dots, 8$, block (i, j) contains the representation matrix for the terms of symmetry i in association type j . Since each symmetry has the form $\iota + \pi$ in one association type, for each i there will be one nonzero block containing the matrix $I + R_\pi^\lambda$. We compute the row canonical form of this matrix; for each λ , the nonzero rows form a basis for the space of identities in degree 11 which are consequences of the alternating property of the ternary commutator. The rank s_λ of this matrix is given in column “sym” of Table 3.

We next construct a $51d_\lambda \times 8d_\lambda$ matrix consisting of $d_\lambda \times d_\lambda$ blocks; the first 43 rows of blocks are the same as in the preceding matrix. For $i = 1, \dots, 8$ and $j = 1, \dots, 8$, block $(43+i, j)$ contains the representation matrix for the terms in association type j of the i -th consequence of $I(a, b, c, d, e, f, g)$. We compute the row canonical form of this matrix; for each λ , the nonzero rows form a basis for the space of identities in degree 11 which are consequences of all the identities of lower degree. The rank sl_λ of this matrix is given in column “sym+lif” of Table 3. In the

#	d_λ	λ	sym	sym+lif	all	new	
1	1	11	8	8	8	0	
2	10	10, 1	80	80	80	0	
3	44	9, 2	352	352	352	0	
4	45	9, 1 ²	360	360	360	0	
5	110	8, 3	880	880	880	0	
6	231	8, 2, 1	1848	1848	1848	0	
7	120	8, 1 ³	960	960	960	0	
8	165	7, 4	1320	1320	1320	0	
10	385	7, 2 ²	3080	3080	3080	0	
12	210	7, 1 ⁴	1680	1680	1680	0	
13	132	6, 5	1056	1056	1056	0	
19	252	6, 1 ⁵	2016	2016	2016	0	
20	330	5 ² , 1	2639	2639	2639	0	
29	210	5, 1 ⁶	1676	1676	1676	0	
40	120	4, 1 ⁷	944	948	948	0	
45	385	3 ² , 1 ⁵	3005	3020	3020	0	
46	330	3, 2 ⁴	2639	2639	2639	0	
49	231	3, 2, 1 ⁶	1764	1795	1795	0	
50	45	3, 1 ⁸	333	349	349	0	
51	132	2 ⁵ , 1	1006	1020	1021	1	←
52	165	2 ⁴ , 1 ³	1242	1269	1270	1	←
53	110	2 ³ , 1 ⁵	807	842	842	0	
54	44	2 ² , 1 ⁷	302	333	333	0	
55	10	2, 1 ⁹	57	76	76	0	
56	1	1 ¹¹	0	7	7	0	

TABLE 3. Representations of S_{11} with dimension < 400

next section we use the name `oldmat`(λ) for the $sl_\lambda \times 8d_\lambda$ matrix in row canonical form containing the nonzero rows.

Finally, we construct a matrix of size $8d_\lambda \times 9d_\lambda$ and use it to find all the identities satisfied by the ternary commutator in degree 11. The first column of $d_\lambda \times d_\lambda$ blocks corresponds to the associative multilinear polynomials, which we identify with the group algebra $\mathbb{Q}S_{11}$. The remaining columns correspond to the eight association types (*). For $i = 1, \dots, 8$ we put the identity matrix in block $(i, i+1)$; in block $(i, 1)$ we put the representation matrix for the expansion of association type i (with the identity permutation of the variables) in the free associative algebra. By the expansion of an association type, we mean the associative polynomial obtained by replacing each occurrence of $[a, b, c]$ by the alternating ternary sum of its arguments; thus each expansion is a sum of $6^5 = 7776$ terms with coefficients ± 1 . In the resulting matrix, the $8d_\lambda \times 8d_\lambda$ submatrix obtained by deleting the first column of blocks is the identity matrix; hence the matrix has rank $8d_\lambda$. We compute the row canonical form of this matrix, and delete the rows whose leading 1s occur within the first d_λ columns. From the remaining matrix, we delete the first d_λ columns, all of whose entries are 0. The result is a matrix of size $a_\lambda \times 8d_\lambda$, with rank a_λ for some $a_\lambda \geq 0$; this number is given in column “all” of Table 3. For each λ , the

(nonzero) rows of this matrix provide a basis for the space of all identities in degree 11 satisfied by the ternary commutator. In the next section we call this matrix $\text{allmat}(\lambda)$.

It is clear that $a_\lambda \geq sl_\lambda$ for every λ : the space of identities which are consequences of identities of lower degree is a subspace of the space of all identities. If $a_\lambda = sl_\lambda$ for some λ then there are no new identities for partition λ . In this case we also need to verify that the two matrices are exactly the same: the first matrix, $\text{oldmat}(\lambda)$ of size $sl_\lambda \times 8d_\lambda$, containing the symmetries of the association types and the consequences of $I(a, b, c, d, e, f, g)$; and the second matrix, $\text{allmat}(\lambda)$ of size $a_\lambda \times 8d_\lambda$, containing all the identities satisfied by the ternary commutator. If $a_\lambda > sl_\lambda$ for some λ then there exist new identities in degree 11 for the representation of S_{11} corresponding to λ . The difference $a_\lambda - sl_\lambda$ is given in column “new” of Table 3.

Owing to the large size of many of the irreducible representations of S_{11} , and the time required to compute the representation matrices R_π^λ , we were able to complete these computations only for the 25 partitions in Table 3, corresponding to the representations with dimensions < 400 , slightly less than half of the total of 56 representations. We found two representations which have new identities: number 51 (dimension 132, partition $2^5 1$) and number 52 (dimension 165, partition $2^4 1^3$). We summarize this discussion in the following theorem.

Theorem 3.1. *New identities in degree 11 for the ternary commutator exist for partitions $2^5 1$ and $2^4 1^3$, and these are the only partitions with corresponding irreducible representations of dimension < 400 which have new identities.*

4. A NEW MULTILINEAR IDENTITY FOR REPRESENTATION 51

Representation 51 is the smaller of the two representations with new identities in Table 3. In this section we obtain an explicit form of a new identity for this representation. (Similar computations could be performed for representation 52.)

From the computations in the previous section we obtain two matrices:

- **oldmat** of size 1020×1056 : this full rank matrix contains the rows representing the symmetries of the association types and the consequences of $I(a, b, c, d, e, f, g)$ for the representation corresponding to partition $\lambda = 2^5 1$.
- **allmat** of size 1021×1056 : this full rank matrix contains the rows representing all the polynomial identities satisfied by the ternary commutator for the representation corresponding to partition $\lambda = 2^5 1$.

The row space of **oldmat** is a subspace of the row space of **allmat**. For a matrix A in row canonical form, we write $\text{leading}(A)$ for the set of column indices for those columns which contain the leading 1 of some row. We have

$$\begin{aligned} \text{leading}(\text{oldmat}) &\subset \text{leading}(\text{allmat}), \\ \text{leading}(\text{allmat}) - \text{leading}(\text{oldmat}) &= \{251\}. \end{aligned}$$

The row of **allmat** which has its leading 1 in column 251 is row 246; this is the row which represents the new identity. This row has 24 nonzero entries, with 16 distinct integer coefficients:

$$-432, -60, -36, -34, -24, -9, 9, 18, 24, 36, 54, 72, 96, 108, 144, 216.$$

The columns of **allmat** correspond to 8 blocks of length $d_\lambda = 132$; the blocks correspond to the association types $(*)$ and the columns in each block correspond

column	t	j	standard tableau									c
251	2	119	1 5	2 6	3 8	4 9	7 11	10				72
253	2	121	1 5	2 6	3 9	4 10	7 11	8				-36
361	3	97	1 4	2 5	3 8	6 10	7 11	9				54
378	3	114	1 5	2 6	3 7	4 8	9 11	10				144
388	3	124	1 5	2 7	3 8	4 10	6 11	9				216
393	3	129	1 6	2 7	3 8	4 10	5 11	9				-60
396	3	132	1 7	2 8	3 9	4 10	5 11	6				36
528	4	132	1 7	2 8	3 9	4 10	5 11	6				9
622	5	94	1 4	2 5	3 7	6 10	8 11	9				108
623	5	95	1 4	2 5	3 8	6 9	7 10	11				-36
626	5	98	1 4	2 5	3 9	6 10	7 11	8				108
645	5	117	1 5	2 6	3 7	4 10	8 11	9				216
653	5	125	1 5	2 7	3 9	4 10	6 11	8				-432
655	5	127	1 6	2 7	3 8	4 9	5 10	11				24
658	5	130	1 6	2 7	3 9	4 10	5 11	8				144
660	5	132	1 7	2 8	3 9	4 10	5 11	6				96
778	6	118	1 5	2 6	3 8	4 9	7 10	11				-24
781	6	121	1 5	2 6	3 9	4 10	7 11	8				72
792	6	132	1 7	2 8	3 9	4 10	5 11	6				-34
890	7	98	1 4	2 5	3 9	6 10	7 11	8				-9
916	7	124	1 5	2 7	3 8	4 10	6 11	9				216
918	7	126	1 5	2 8	3 9	4 10	6 11	7				108
924	7	132	1 7	2 8	3 9	4 10	5 11	6				108
1056	8	132	1 7	2 8	3 9	4 10	5 11	6				18

TABLE 4. Row 246 of `allmat` representing the new identity

to the standard tableaux for partition $\lambda = 2^5 1$ in lexicographical order:

1	2	1	2	1	2	...	1	6	1	7
3	4	3	4	3	4		2	8	2	8
5	6	5	6	5	6		3	9	3	9
7	8	7	8	7	9		4	10	4	10
9	10	9	11	8	10		5	11	5	11
11		10		11			7		6	

Table 4 gives complete information about the row representing the new identity, where t is the association type, j is the tableau index, and c is the coefficient; the standard tableaux are given in flattened form as a sequence of rows.

To convert this data into an explicit identity for the ternary commutator, we use the correspondence between matrix units in the representation matrices and elements of the group algebra [4, Remark 2, p. 2004]. We summarize this result in the general case. Given a partition λ of n , let d_λ be the dimension of the corresponding irreducible representation of S_n . For $1 \leq i, j \leq d_\lambda$ we construct the element of the group algebra $\mathbb{Q}S_n$ corresponding to the matrix unit E_{ij}^λ under the isomorphism of $\mathbb{Q}S_n$ with a direct sum of full matrix algebras. Let $T_1, \dots, T_{d_\lambda}$ be the standard tableaux for λ in lexicographical order. For each $i = 1, \dots, d_\lambda$

let R_i (respectively C_i) be the subgroup of S_n which leaves the rows (respectively columns) of T_i fixed as sets. For $i, j = 1, \dots, d_\lambda$ let s_{ij} be the permutation for which $s_{ij}T_i = T_j$. We define elements $D_{ij} \in S_n$ as follows:

$$D_{ii} = \frac{d_\lambda}{n!} \sum_{\sigma \in R_i} \sum_{\tau \in C_i} \epsilon(\tau) \sigma \tau, \quad D_{ij} = D_{ii} s_{ij}^{-1}.$$

These elements in general do not satisfy the multiplication formulas for matrix units. To obtain the matrix units, let A_π^λ be the matrix defined by Clifton [7] for the permutation π . For the identity permutation ι , the matrix A_ι^λ is not necessarily the identity matrix, but it is always invertible. Let (a_{ij}) be the inverse matrix $(A_\iota^\lambda)^{-1}$; then the element of $\mathbb{Q}S_n$ corresponding to the matrix unit E_{ij}^λ is

$$E_{ij}^\lambda \longleftrightarrow \sum_{k=1}^d a_{jk} D_{ik}.$$

We then have the required relations $E_{ij}^\lambda E_{kl}^\lambda = \delta_{jk} E_{il}^\lambda$.

We now return to our discussion of the new identity in degree 11 for the ternary commutator. Since we are dealing with a single identity we may assume that $i = 1$: any row of the representation matrix can be moved to row 1 by left multiplication by an element of the group algebra. Moreover, we need to consider only those values of j which appear in Table 4:

$$j = 94, 95, 97, 98, 114, 117, 118, 119, 121, 124, 125, 126, 127, 129, 130, 132.$$

We compute the matrix A_ι^λ and find that it has the form $I + U$ where U is a strictly upper triangular matrix with 262 nonzero entries from the set $\{\pm 1\}$. The inverse matrix $(A_\iota^\lambda)^{-1}$ has the form $I + V$ where V is a strictly upper triangular matrix with 424 nonzero entries from the set $\{\pm 1, \pm 2\}$. For all except one of the values of j listed above, the corresponding row of $(A_\iota^\lambda)^{-1}$ has only one nonzero entry, which is the diagonal entry 1. Therefore, for all these values except $j = 118$ the matrix unit is $E_{1j} = D_{1j}$; the exceptional case is $E_{1,118} \leftrightarrow D_{1,118} - D_{1,126} + D_{1,131}$. It remains to apply the association types $(*)$ to the elements of the group algebra. Let $f \in \mathbb{Q}S_{11}$ be arbitrary, and let $t = 1, \dots, 8$ be one of the association types. We regard f as a multilinear associative polynomial in the variables a_1, \dots, a_{11} . We convert this into a polynomial in the ternary commutator by applying association type t to every monomial. The resulting element of the free alternating ternary algebra will be denoted $[f]_t$. We can now write down the new identity.

Theorem 4.1. *The following multilinear polynomial identity in degree 11 is satisfied by the ternary commutator and does not follow from the symmetries of the association types and the consequences of $I(a, b, c, d, e, f, g)$:*

$$\begin{aligned} & 72 [D_{1,119}]_2 - 36 [D_{1,121}]_2 + 54 [D_{1,97}]_3 + 144 [D_{1,114}]_3 + 216 [D_{1,124}]_3 \\ & - 60 [D_{1,129}]_3 + 36 [D_{1,132}]_3 + 9 [D_{1,132}]_4 + 108 [D_{1,94}]_5 - 36 [D_{1,95}]_5 \\ & + 108 [D_{1,98}]_5 + 216 [D_{1,117}]_5 - 432 [D_{1,125}]_5 + 24 [D_{1,127}]_5 + 144 [D_{1,130}]_5 \\ & + 96 [D_{1,132}]_5 - 24 ([D_{1,118}]_6 - [D_{1,126}]_6 + [D_{1,131}]_6) + 72 [D_{1,121}]_6 \\ & - 34 [D_{1,132}]_6 - 9 [D_{1,98}]_7 + 216 [D_{1,124}]_7 + 108 [D_{1,126}]_7 + 108 [D_{1,132}]_7 \\ & + 18 [D_{1,132}]_8. \end{aligned}$$

This identity implies all the new identities for partition $2^5 1$.

5. A NEW NON-MULTILINEAR IDENTITY FOR REPRESENTATION 51

For partition $\lambda = 2^5 1$ we expect that there will be a new identity in which each term consists of a permutation of the multiset $a^2 b^2 c^2 d^2 e^2 f$ with one of the eight association types (*). The total number of such permutations is $\binom{11}{2,2,2,2,2,1} = 1247400$. These permutations will be called “associative monomials”; they form a basis of the homogeneous subspace A_δ of the free associative algebra on six generators with multidegree $\delta = (2, 2, 2, 2, 2, 1)$. The total number of alternating ternary monomials in each association type can be determined by direct enumeration. If $p_1 \cdots p_{11}$ denotes an associative monomial, then the eight association types require the following conditions, where $<$ denotes lexicographical order:

$$(**) \quad \left\{ \begin{array}{ll} 1: & p_1 < p_2 < p_3, \quad p_4 < p_5, \quad p_6 < p_7, \quad p_8 < p_9, \quad p_{10} < p_{11} \\ 2: & p_1 < p_2 < p_3, \quad p_4 < p_5 < p_6, \quad p_8 < p_9, \quad p_{10} < p_{11}, \\ & p_1 p_2 p_3 < p_4 p_5 p_6 \\ 3: & p_1 < p_2 < p_3, \quad p_4 < p_5, \quad p_6 < p_7 < p_8, \quad p_{10} < p_{11} \\ 4: & p_1 < p_2 < p_3, \quad p_4 < p_5 < p_6, \quad p_7 < p_8 < p_9, \quad p_{10} < p_{11}, \\ & p_1 p_2 p_3 < p_4 p_5 p_6 < p_7 p_8 p_9 \\ 5: & p_1 < p_2 < p_3, \quad p_4 < p_5, \quad p_6 < p_7, \quad p_8 < p_9 < p_{10} \\ 6: & p_1 < p_2 < p_3, \quad p_4 < p_5 < p_6, \quad p_8 < p_9 < p_{10}, \\ & p_1 p_2 p_3 < p_4 p_5 p_6 \\ 7: & p_1 < p_2 < p_3, \quad p_4 < p_5, \quad p_6 < p_7 < p_8, \quad p_9 < p_{10}, \\ & p_1 p_2 p_3 p_4 p_5 < p_6 p_7 p_8 p_9 p_{10} \\ 8: & p_1 < p_2 < p_3, \quad p_4 < p_5, \quad p_6 < p_7 < p_8, \quad p_9 < p_{10} < p_{11}, \\ & p_6 p_7 p_8 < p_9 p_{10} p_{11} \end{array} \right.$$

The total number of permutations satisfying these conditions is

$$6720 + 1980 + 4010 + 180 + 4010 + 1190 + 2000 + 550 = 20640.$$

The resulting bracketed permutations will be called “nonassociative monomials”; they form a basis of the homogeneous subspace N_δ of the free alternating ternary algebra on six generators with multidegree δ .

At this point, we would like to construct a matrix of size 1247400×20640 in which the (i, j) entry is the coefficient of the i -th associative monomial in the expansion of the j -th nonassociative monomial; as before, by expansion we mean repeated application of the alternating ternary sum. This matrix represents the “expansion map” $E_\delta: N_\delta \rightarrow A_\delta$ with respect to the bases of associative and nonassociative monomials. The polynomial identities satisfied by the ternary commutator are the (nonzero) vectors in the kernel K_δ of this linear map. The matrix representing E_δ is very sparse, since each expansion contains only 7776 terms; more than 99% of the entries are 0. However, processing a matrix of this size is not practical. We therefore begin by storing the expansions of the nonassociative monomials in a matrix of size 7776×20640 ; the (i, j) entry contains the i -th term of the j -th expansion in the form $\pm k$. The sign ± 1 is the coefficient of the term and the absolute value k is the lexicographical index of the associative monomial.

We now observe that $1247400 = 77 \cdot 16200$. We construct a matrix with an upper block of size 20640×20640 and a lower block of size 16200×20640 , and initialize it to zero. We then perform the following iteration for $\ell = 1, \dots, 77$:

- For each column index j , extract the terms of the corresponding expansion whose indices k lie in the range $16200(\ell-1) < k \leq 16200\ell$.
- Store the corresponding coefficients in the appropriate row of the lower block; index k goes to row $k-16200(\ell-1)$.
- After all the columns have been processed, and the lower block has been filled, compute the row canonical form. (The lower block is now zero.)

At the end of this iteration, the nullspace of the matrix contains the coefficient vectors of the polynomial identities satisfied by the ternary commutator. The rank of the matrix is 19964, and so the nullity is 676. We compute the canonical basis of the nullspace from the row canonical form by setting the free variables equal to the standard basis vectors in dimension 676 and solving for the leading variables. We summarize this discussion in the following lemma.

Lemma 5.1. *The kernel K_δ of the linear map $E_\delta: N_\delta \rightarrow A_\delta$ has dimension 676.*

The next step is to determine the subspace $L_\delta \subset K_\delta$ consisting of the polynomial identities which are consequences of the known identity $I(a, b, c, d, e, f, g)$ in degree 7. (The consequences of the alternating properties in degree 3 have already been excluded by our choice (**) of nonassociative monomials.) For each consequence of $I(a, b, c, d, e, f, g)$ in Table 2, we must determine the corresponding substitutions of the variables $a^2b^2c^2d^2e^2f$, recalling that $[a, b, c]$ alternates in all three arguments and $I(a, b, c, d, e, f, g)$ alternates in b, c, d, e, f, g . If $q_1 \cdots q_{11}$ denotes an associative monomial, then the eight consequences require the following conditions, where $<$ denotes lexicographical order:

- 1: $q_1 < q_{10} < q_{11}, \quad q_8 < q_9, \quad q_2 < q_3 < q_4 < q_5 < q_6 < q_7$
- 2: $q_1 < q_8 < q_9, \quad q_2 < q_{10} < q_{11}, \quad q_3 < q_4 < q_5 < q_6 < q_7$
- 3: $q_1 < q_8 < q_9, \quad q_2 < q_3 < q_4 < q_5 < q_6 < q_7, \quad q_{10} < q_{11}$
- 4: $q_2 < q_{10} < q_{11}, \quad q_8 < q_9, \quad q_3 < q_4 < q_5 < q_6 < q_7$
- 5: $q_2 < q_8 < q_9, \quad q_3 < q_{10} < q_{11}, \quad q_4 < q_5 < q_6 < q_7, \quad q_2q_8q_9 < q_3q_{10}q_{11}$
- 6: $q_2 < q_8 < q_9, \quad q_3 < q_4 < q_5 < q_6 < q_7, \quad q_{10} < q_{11}$
- 7: $q_2 < q_3 < q_4 < q_5 < q_6 < q_7, \quad q_8 < q_{10} < q_{11}$
- 8: $q_2 < q_3 < q_4 < q_5 < q_6 < q_7, \quad q_8 < q_9, \quad q_{10} < q_{11}$

The total number of substitutions satisfying these conditions is

$$10 + 50 + 10 + 170 + 215 + 170 + 20 + 30 = 675.$$

We note that this number is exactly one less than the dimension of K_δ . Each of these consequences of $I(a, b, c, d, e, f, g)$ expands to a linear combination of terms which consist of a sign and a permutation of $a^2b^2c^2d^2e^2f$ with one of the eight association types (*). We construct a matrix M of size 676×20640 and fill the first 675 rows with the coefficient vectors of these consequences of $I(a, b, c, d, e, f, g)$. We compute the row canonical form, and find that the rank is 675, so these substitutions are linearly independent. We summarize this discussion in the following lemma.

Lemma 5.2. *The subspace $L_\delta \subset K_\delta$ spanned by the consequences of the identity $I(a, b, c, d, e, f, g)$ has dimension 675.*

It follows that any complementary subspace to L_δ inside K_δ has dimension 1; this agrees with the result for partition 2^51 from Table 3. Our next task is to find a basis for this complementary subspace.

In each vector of the canonical basis of K_δ , we clear denominators and cancel common factors, so that the components are relatively prime integers. We then sort the vectors by increasing Euclidean length; the minimum square length is 60 and the maximum is 79134357. In these 676 vectors, the minimum number of nonzero components is 58 and the maximum is 15901; the minimum number of distinct coefficients is 2 and the maximum is 509. We copy these row vectors one at a time to the last row of the matrix M ; after each row, we reduce the matrix. The vector which increases the rank from 675 to 676 corresponds to item 585 in the sorted list, which is item 241 of the original (unsorted) canonical basis of the nullspace. This vector has 10292 nonzero components.

Theorem 5.3. *There exists a non-multilinear polynomial identity in degree 11 satisfied by the ternary commutator which is not a consequence of the identities of lower degree. This identity has 10292 terms in the six variables $a^2b^2c^2d^2e^2f$; it involves only association types 1, 2, 3, 5, 6 and its coefficients are $\pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 6, \pm 7, \pm 8, \pm 9, \pm 10, \pm 11, -12, 13$.*

To complete the computational verification of this theorem, we proceed as follows. We first check by direct expansion that this polynomial is in fact an identity satisfied by the ternary commutator. We create a vector of length 1247400, initialized to 0; this vector represents a linear combination of the 1247400 associative monomials in the variables $a^2b^2c^2d^2e^2f$. We then expand each of the 10292 terms of the polynomial using the alternating ternary sum; for each term we obtain a linear combination of 7776 terms, where each term is (\pm) one of the associative monomials. For each term in the expansion, we add the appropriate coefficient to the corresponding component of the array. After all terms of the polynomial have been expanded and added to the vector, every component of the vector is 0.

Finally, we need to verify that this polynomial increases the rank from 1020 to 1021 in the irreducible representation for partition 2^51 ; see Table 3. We first linearize the polynomial: each term $\cdots a \cdots a \cdots$ has two occurrences of the variable a and produces two terms $\cdots a \cdots g \cdots$ and $\cdots g \cdots a \cdots$; then we similarly replace bb by bh and hb , cc by ci and ic , dd by dj and jd , ee by ek and ke . Each term of the original non-multilinear polynomial produces 32 terms in the linearized polynomial, giving a total of $32 \cdot 10292 = 329344$ terms.

We now use the representation theory of the symmetric group as described in Section 4 to redo the computation for partition 2^51 with dimension $d_\lambda = 132$. We construct a $52d_\lambda \times 8d_\lambda$ matrix consisting of $d_\lambda \times d_\lambda$ blocks; the first 43 rows of blocks contain the representation matrices for the symmetries of the association types, and the next 8 rows of blocks contain the representation matrices for the consequences of $I(a, b, c, d, e, f, g)$. The last row of blocks contains the representation matrices for the linearized form of the new identity. We compute the row canonical form of this matrix and find that its rank is 1021, as required. Furthermore, we verify that the nonzero rows of this matrix coincide exactly with the 1021×1056 matrix `allmat` obtained from the expansions of the association types.

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